Heterogeneous morphological granulometries

Sinan Batman, Edward R. Dougherty,*, Francis Sand

*Department of Electrical Engineering, Johns Hopkins University, USA
bDepartment of Electrical Engineering, Texas Center for Applied Technology, Texas A&M University, 214 Wisenbaker Engineering Research Center, College station, TX 77843-3407, USA
#School of Computer Science and Information Systems, Fairleigh Dickinson University, USA

Received 18 January 1999; received in revised form 2 May 1999; accepted 23 June 1999

Abstract

The most basic class of binary granulometries is composed of unions of openings by structuring elements that are homogeneously scaled by a single parameter. These univariate granulometries have previously been extended to multivariate granulometries in which each structuring element is scaled by an individual parameter. This paper introduces the more general class of filters in which each structuring element is scaled by a function of its sizing parameter, the result being multivariate heterogeneous granulometries. Owing to computational considerations, of particular importance are the univariate heterogeneous granulometries, for which scaling is by functions of a single variable. The basic morphological properties of heterogeneous granulometries are given, analytic and geometric relationships between multivariate and univariate heterogeneous pattern spectra are explored, and application to texture classification is discussed. The homogeneous granulometric mixing theory, both the representation of granulometric moments and the asymptotic theory concerning the distributions of granulometric moments, is extended to heterogeneous scaling. © 2000 Pattern Recognition Society. Published by Elsevier Science Ltd. All rights reserved.

Keywords: Granulometry; Mathematical morphology; Mixing theory; Pattern spectrum; Texture

1. Introduction

Granulometries are parameterized families of morphological openings that are used for granular filtering (clutter removal) [1–5] and pattern and texture classification [6–12]. As originally conceived, the most basic type of granulometry is formed as a union of binary openings in which each structuring element is a homothetic $tB$ and the scaling (sizing) parameter $t$ is the same for all structuring elements [13]. There are various extensions of this concept, the most relevant to the present paper being multivariate granulometries, in which the structuring elements are scaled by independent individual parameters [14]. From the standpoint of texture classification, allowing each structuring element to be scaled independently provides a more general class of filters and thereby facilitates increased discriminatory power. The present paper takes this form of generalization one step further by having each structuring element scaled by an increasing function of its sizing parameter. This heterogeneous scaling produces a larger class of filter families and, by using a single parameter with individual scaling functions, we can obtain sieving filters that are parameterized along paths in multidimensional space. Sieving along a path imposes no extra computational cost than that which occurs with a classical univariate granulometry; however, it can lead to improved classification over homogeneously scaled univariate granulometries without the growing computational burden commensurate with using multivariate granulometries. After defining the class of heterogeneous granulometries and giving some basic morphological properties, analytic and geometric relationships between multivariate and univariate heterogeneous pattern spectra are explored, and application to...
texture classification is discussed. Morphological texture classification often involves feature vectors whose components are moments of granulometric pattern spectra. Finally, we extend the granulometric mixing theorems to heterogeneous granulometries. These theorems provide representations for granulometric moments for classes of disjoint randomly sized granular images and provide asymptotic distributions for these moments [14–17]. Before proceeding, we review some basic definitions.

A granulometry is a family \( \{\Psi_t\} \), \( t > 0 \), such that \( \Psi_t \) is antiextensive \( \{\Psi_t(A) \subset A\} \); \( \Psi_t \) is increasing \( \{A \subset B \implies \Psi_t(A) \subset \Psi_t(B)\} \); for all \( u, v > 0 \), \( \Psi_{uv} = \Psi_v \Psi_u = \Psi_{\max(u,v)} \); and \( \Psi_t \) is translation invariant \( \{\Psi_t(A + x) = \Psi_t(A) + x\} \). For completeness, for \( t = 0 \), we define \( \Psi_0(A) = A \). \( \{\Psi_t\} \) is a Euclidean granulometry if, for any \( t > 0 \) and any image \( A \), \( \Psi_t(A) = t \Psi_1[(1/t)A] \). For \( u < v \), \( \Psi_u \subset \Psi_v \). The basic representation of Matheron states that \( \{\Psi_t\} \) is a Euclidean granulometry if and only if there exists a set family \( \mathcal{A} = \{B_i\} \), called the generator of the granulometry, such that

\[
\Psi_t(S) = \bigcup_{i} S \circ rB_i, \tag{1}
\]

where the opening \( S \circ B \) is the union of all translates of \( B \) that are subsets of \( S \) [13].

Given two sets \( A \) and \( B \), \( A \) is \( B \)-open (open relative to \( B \)) if \( A \circ B = A \). If \( A \) is \( B \)-open, then \( S \circ A \subset S \circ B \) for any set \( S \). For a convex set \( B \) and \( r \geq t \), \( rB \) is \( tB \)-open. Hence, \( S \circ rB \subset S \circ tB \). Thus, if the generator sets are convex, then the double union reduces to the single union

\[
\Psi_t(S) = \bigcup_i S \circ tB_i \tag{2}
\]

For compact generator sets, the double union reduces to a single union if and only if all generator sets are convex [13]. Owing to the difficulty of forming the union over all \( r \geq t \), applications involve finite numbers of convex, compact structuring elements and the single-union formulation.

\( \{\Psi_t(S)\} \) is decreasing for increasing \( t \) and if \( S \) is compact, then \( \Psi_t(S) = \emptyset \) for sufficiently large \( t \). Letting \( \alpha \) denote Lebesgue measure, \( \Omega(t) = \alpha[\Psi_t(S)] \) is a decreasing function of \( t \). The normalization \( \Phi(t) = 1 - \Omega(t)/\alpha[S] \) is a probability distribution function and the derivative \( \Phi(t) \) is a probability density called the pattern spectrum of \( S \). The moments of \( \Phi(t) \) are used as texture features. Because \( S \) is modeled as a random set, \( \Phi(t) \) is a random function and its moments are random variables possessing probability distributions dependent on \( S \).

2. Heterogeneous granulometries

Extending Eq. (2), this paper treats filter families formed by finite unions of openings by scaled structuring elements in which the scalings are functions of individual parameters. In this sense, for a family \( \mathcal{A} = \{B_1, B_2, \ldots, B_n\} \) of convex, compact sets, we define a heterogeneous multivariate granulometry to be a family of filters defined by

\[
\Psi_t(S) = \bigcup_{i=1}^n S \circ h_i(t_i)B_i, \tag{3}
\]

where \( t = (t_1, t_2, \ldots, t_n), t_i > 0 \) for \( i = 1, 2, \ldots, n \), and \( h_1, h_2, \ldots, h_n \) are strictly increasing continuous functions of \( t_1, t_2, \ldots, t_n \), respectively, on \([0, \infty)\); such that \( h_i(0) = 0 \) and \( h_i(t_i) \to \infty \) as \( i \to \infty \). For any \( t = (t_1, t_2, \ldots, t_n) \) for which there exists \( t_i = 0 \), define \( \Psi_t(S) = S \). Because \( h_i \) is strictly increasing and continuous, it represents an increasing bijection. Hence, \( h_i(t)B_i \) is \( h_i(t) \)-open for \( r \geq t \) for all \( i \). To avoid redundancy, \( B_1, B_2, \ldots, B_n \) are assumed to be distinct shapes relative to scalar multiplication: namely, if \( i \neq j \), then there does not exist a scalar \( s \) such that \( sB_i = B_j \).

Letting \( h = (h_1, h_2, \ldots, h_n) \), we call \( \{\Psi_t\} \) a \( (\mathcal{A}, h, t) \)-granulometry, and \( \mathcal{A} \) is the generator of the granulometry. \( \Psi_t \) is a \( t \)-opening [increasing, idempotent, antiextensive, and translation invariant] with base \( [h_1(t_i)B_1, \ldots, h_n(t_n)B_n] \). The invariant class of \( \Psi_t \), \( \text{Inv}[\Psi_t] \), is the collection of all sets \( S \) such that \( \Psi_t(S) = S \). Because \( h_1, h_2, \ldots, h_n \) are strictly increasing, if \( t \geq s \), meaning \( t_i \geq s_i \) for \( i = 1, 2, \ldots, n \), then \( \Psi_t \subset \Psi_s \) and \( \text{Inv}[\Psi_t] \subset \text{Inv}[\Psi_s] \). If \( h_1, h_2, \ldots, h_n \) all equal the identity function, then we obtain the previously studied multivariate granulometries [14], which we now term homogeneous multivariate granulometries, or \( (\mathcal{A}, h) \)-granulometries. If we let \( k \to \infty \) for some fixed \( k \) and \( S \) is compact, then \( S \circ h_k(t_k) \to \emptyset \) and we obtain the marginal granulometry given by the union of Eq. (3) with the \( k \)th term deleted.

If, for \( r > 0 \), we fix \( t = (t_1, t_2, \ldots, t_n) \) and define

\[
\Lambda_r(S) = \bigcup_{i=1}^n S \circ rh_i(t_i)B_i, \tag{4}
\]

then \( \{\Lambda_r\} \) is a Euclidean granulometry with generator \( [h_1(t_1)B_1, \ldots, h_n(t_n)B_n] \) and the original Euclidean theory yields the following properties: if \( r \geq s > 0 \), then \( \text{Inv}[\Lambda_r] \subset \text{Inv}[\Lambda_s] \), \( \Lambda_r \Lambda_r = \Lambda_0 \Lambda_r = \Lambda_{r+s} \); and \( \text{Inv}[\Lambda_r] = r \text{Inv}[\Lambda_r] \).

If we let \( t = (t, t, \ldots, t) \), then Eq. (3) yields a univariate heterogenous granulometry \( \{\Psi_t\} \), which we term a \( (\mathcal{A}, h) \)-granulometry. Because each \( \Psi_t \) is a \( t \)-opening and for \( t \geq s \), \( \text{Inv}[\Psi_t] \subset \text{Inv}[\Psi_s] \), \( \{\Psi_t\} \) is a granulometry. If we let \( h_i(t_1) = t \) for all \( i \), then the \( (\mathcal{A}, h) \)-granulometry is a Euclidean granulometry with generator \( \mathcal{A} \), which in the present context we will call a \( \mathcal{A} \)-granulometry. If each function \( h_i \) is linear, say \( h_i(t) = a_i t \), then grouping \( a_i \) with \( B_i \) in \( a_i B_i \) shows \( \{\Psi_t\} \) to be the Euclidean granulometry with generator \( \{a_1 B_1, \ldots, a_n B_n\} \).

A \( (\mathcal{A}, h) \)-granulometry is an upper semicontinuous (u.s.c.) granulometry. To demonstrate this, we need to show that the mapping \( (t, S) \to \Psi_t(S) \) is u.s.c. on \( \mathbb{R}^+ \times \mathcal{F} \).
where $\mathcal{N}$ is the space of compact sets. The homothetic
$(t, B) \mapsto tB$ is a continuous mapping on $\mathcal{N} \times \mathcal{N}$
and therefore $(t, B) \mapsto h(t)B$ is continuous when $h(t)$ is continuous
and positive valued. Consequently, $(t, B) \mapsto h_i(t)B_i$ is
continuous for each $i$. Since opening is u.s.c., this implies that $S \mapsto h_i(t)B_i$ is u.s.c. for all $i$, and, since union is continuous,
$\Psi_i$ is u.s.c. For the Euclidean case, $h_i(t)B_i = tB_i$.

In the multivariate setting, the size distribution and pattern spectrum are
defined by $\Omega(t) = \pi[\Psi_i(S)]$ and $\Phi(t) = 1 - (1 - \psi(t)/\pi[S])$, respectively. It has been shown that $\Phi(t)$ is a probability distribution function for homogene-
ous multivariate granulometries [14]. A similar argument shows that $\Phi(t)$ is a probability distribution function for heterogeneous multivariate granulometries.

3. Univariate heterogeneous granulometric size distributions

Despite the large amount of information extracted by multivariate granulometries, a univariate approach is
computationally attractive. For a $(\mathcal{B}, \mathcal{h})$-granulometry,
the size distribution is a function, $\Omega(t)$, of a single variable. Let $\gamma$ be the curve defined by $h(t) = (h_1(t), \ldots, h_n(t))$ and $s = \xi(t)$ be the arc length of $\gamma$. Because $\mathcal{h}$ is a bijection,
$\xi(t)$ is strictly increasing. Hence, the size distribution can be viewed as a function of $h(t)$ or of $s$. Rigorously,
$\Omega(t) = \Omega_s(\xi(t))$, where $\Omega_1 = \Omega_s^{-1}$, and $\Omega(t) = \Omega_2(h(t))$, where $\Omega_2 = \Omega h^{-1}$. Similar variable changes apply to
the pattern spectrum and the derivative of $\Phi(t)$ can be obtained via the chain rule. Relative to arc length,

$$
\Phi(t) = \Phi_1(\xi(t)) = 1 - \Omega_1(\xi(t))/\pi[S],
$$

$$
\frac{d\Phi}{dt} = \frac{d\Phi_1}{ds} \frac{ds}{dt} = - \frac{1}{\pi[S]} \frac{d\Omega_1}{ds} \frac{ds}{dt}.
$$

Relative to $h(t)$,

$$
\Phi(t) = \Phi_2(h(t)) = 1 - \Omega_2(h(t))/\pi[S],
$$

$$
\frac{d\Phi}{dt} = \sum_{k=1}^{n} \frac{\partial \Phi_2}{\partial h_k} \frac{dh_k}{dt} = - \frac{1}{\pi[S]} \sum_{k=1}^{n} \frac{\partial \Omega_2}{\partial h_k} \frac{dh_k}{dt}.
$$

The pattern spectrum can be viewed as a function of arc length $s$ according to $\Phi_1(s) = \Phi s^{-1}(s)$ and the chain-rule differentia-
tion

$$
\frac{d\Phi_1}{ds} = \sum_{i=1}^{n} \frac{\partial \Phi_2}{\partial h_i} \frac{dh_i}{ds} = \left[ \sum_{i=1}^{n} \left( \frac{dh_i}{ds} \right)^2 \right]^{-1/2} \sum_{i=1}^{n} \frac{\partial \Phi_2}{\partial h_i} \frac{dh_i}{ds},
$$

$$
\Phi_1(s)/ds \text{ can be derived from } \Phi(t) \text{ by means of the preceding formula together with the formula}
$$

$$
\Phi_2(h(t)) = \int_0^{h(t)} \cdots \int_0^{h(t)} \Phi_2(\tau_1, \tau_2, \ldots, \tau_n),
$$

where $\Phi_2 = \Phi h^{-1}$. These expressions are useful because a key concern is finding univariate heterogeneous
granulometries that provide texture discrimination that is almost as good as that provided by a multivariate
homogeneous granulometry. Eqs. (9) and (10) allow derivation of pattern spectra for the former from pattern
spectra of the latter when univariate heterogeneous granulometries are viewed as functions of arc length.

Since granulometries are employed to extract geometric information, it is important to gain an appreciation of
the relationship between a homogeneous multivariate
granulometry with generator $\mathcal{B}$ and heterogeneous
univariate granulometries possessing the same generator.
In particular, we would like to consider the relationship
between their pattern spectra. To place the geometric
analysis in $\mathcal{N}^3$, so that we can picture it, we consider the
case of a disjoint union

$$
S = (B_1 + z_1) \cup (B_2 + z_2),
$$

formed by translations of $B_1$ and $B_2$. To give the size
distribution a structured geometry, we will assume that $B_1$ and $B_2$ are orthogonal [15], which means that

$$
(S \circ tB_1) \cup (S \circ tB_2) = \begin{cases} S & \text{if } t \leq 1, \\
\emptyset & \text{if } t > 1. 
\end{cases}
$$

We apply the homogeneous multivariate granulometry with generator $\mathcal{B} = \{B_1, B_2\}$. Let $H(t) = 1$ if $t > 0$ and
$H(t) = 0$ if $t \leq 0$, and $\bar{H}(t) = 1$ if $t \geq 0$ and $\bar{H}(t) = 0$ if $t < 0$. Then the bivariate size distribution, normalized
bivariate size distribution, and bivariate pattern spectrum
are given by

$$
\Omega(t_1, t_2) = \pi[S]H(1 - t_1)\bar{H}(1 - t_2)
$$

$$
+ \pi[S \circ tB_1]H(t_2 - 1)
$$

$$
+ \pi[S \circ tB_2]H(t_1 - 1),
$$

$$
\Phi(t_1, t_2) = \Phi(t_1)H(t_2 - 1) + \Phi(t_2)H(t_1 - 1)
$$

$$
- H(t_1 - 1)H(t_2 - 1),
$$

$$
\Phi(t_1, t_2) = \frac{\partial \Phi_1(t_1, t_2)}{\partial t_1 \partial t_2} = \Phi_1(t_1)\delta(t_2 - 1)
$$

$$
+ \Phi_1(t_2)\delta(t_1 - 1) - \delta(t_1 - 1)\delta(t_2 - 1),
$$

respectively. These functions are illustrated in Figs. 1–3,
where $t_{12}(t_{23})$ is the largest scalar $t$ for which $\pi[S \circ tB_1] \neq 0$ ($\pi[S \circ tB_2] \neq 0$).

Now consider a more general situation in which $S$ is
a disjoint union of translated scalar multiples of $B_1$ and $B_2$ of the form

$$
S = \bigcup_{k=1}^{n} (t_k B_1 + z_{1,k}) \cup (t_k B_2 + z_{2,k}).
$$

The bivariate pattern spectrum, which is the generalized
second-order mixed partial derivative of the normalized
The bivariate size distribution, has the form given in Fig. 4 (excluding the lines $\Gamma, \Gamma_{\text{max}}, \Gamma_{\text{min}}$, and the shaded region $\Lambda$). Were we to apply the univariate homogeneous $\mathcal{B}$-granulometry, its path would be the $45^\circ$ line running through the triangles in the figure. This $\mathcal{B}$-granulometry will simultaneously eliminate pairs of grains $t_1B_1 + z_{1,k}$ and $t_2B_2 + z_{2,k}$. Its size distribution is a step function with a step of size $t_2(z[B_1] + z[B_2])$ at each $t_k$. So long as the path of any heterogeneous univariate granulometry remains between $\Gamma_{\text{min}}$ and $\Gamma_{\text{max}}$, it too will have a size distribution that is a step function; however, same sized grains will not be simultaneously eliminated and therefore there will be more steps corresponding to elimination of individual grains.

4. Mixing theory

The homogeneous granulometric mixing theorems provide representations for granulometric moments of certain disjoint grain processes and asymptotic distributions for these moments. The mixing theory has been used to estimate parameters of grain-size distributions in random grain processes and to estimate the proportions of differing grain types. The next theorem, whose proof is in the appendix, provides a heterogeneous multivariate granulometric mixing theorem.

**Theorem 1.** Let $S$ be composed of randomly sized, disjoint translates arising from $d$ compact sets $A_1, A_2, \ldots, A_d,$

$$S = \bigcup_{i=1}^{d} \bigcup_{j=1}^{m_i} r_{ij}A_i + x_{ij},$$

and $h(t) = (t_1^{p_1}, t_2^{p_2}, \ldots, t_n^{p_n})$, with $p_i$ an integer for $i = 1, 2, \ldots, n$. The granulometric moments of the $(\mathcal{B}, h, t)$-granulometry of Eq. (3) are given by

$$\mu^{(h)}(S) = \frac{\sum_{i=1}^{d} z[A_i] \mu^{(v)}(A_i) \sum_{j=1}^{m_i} s_{ij}^{p_i}}{\sum_{i=1}^{d} z[A_i] \sum_{j=1}^{m_i} s_{ij}^{p_i}},$$

where $\mu^{(v)}(A_i)$ is a step function with a step of size $z[A_i]$ at each $t[i].$
\[ k = k_1 + k_2 + \cdots + k_n, \quad P = p_1 p_2 \cdots p_n, \quad s_{ij} = r_{ij}^P, \]

\[
\begin{align*}
\frac{v}{w} &= \sum_{i=1}^{n} k_i + 1 - p_i, \\
L &= 2 + \sum_{i=1}^{n} k_i + 1}))) P.
\end{align*}
\]

The Taylor expansion of \( s_{ij} \) about \( \bar{s}_i \), the sample mean of \( s_{i1}, s_{i2}, \ldots, s_{im} \), is given by

\[ s_{ij} = L_{ij} \bar{s}_i - \bar{s}_i, \]

where \( \bar{s} = E[s_i] \). Therefore Eq. (18) can be rewritten as

\[
\mu^{(b)}(S) = \sum_{i=1}^{d} \sum_{j=1}^{m} \sum_{c=0}^{\mu^{(b)}} \left( \frac{L}{c} \right) 3^{i-1(\bar{s}_i - \bar{s})},
\]

where the \( m_{ec} \) is the \( c \)th sample central moment of \( s_{i1}, s_{i2}, \ldots, s_{im} \).

The representation for \( \mu^{(b)}(S) \) in Eq. (18) is similar in form to the homogeneous multivariate granulometric mixing theorem for the model of Eq. (17) \[14\]. The main difference is that \( \mu^{(b)}(A_i) \) is a fractional moment of the related granulometry. An important difference with the homogeneous theory is that the Taylor series expansion is evaluated around the mean of the transformed variable \( s_i \), not \( r_i \). Without such a change of variables, the expansion would have infinite terms and it would be difficult to discuss the normality of the moments. The family of heterogeneous paths for which Theorem 1 is applicable is larger than the simple monomials \( r_{ij} \), since it depends only on some properties of these monomials; however, we see no practical benefit in stating the theorem more generally in terms of these properties.

If \( S \) is a random set of the form

\[ S = \sum_{j=1}^{m} r_j A \]  

for which the scalars \( r_1, r_2, \ldots, r_m \) are independent and identically distributed, then the discussions of Refs. \[15,16\] apply and we can apply a theorem of Cramér \[18\] to conclude that \( \mu^{(b)}(S) \) is asymptotically normal (as \( m \to \infty \)) and obtain asymptotic expressions for the statistical moments of granulometric moments \( \mu^{(b)}(S) \). It can be shown that the expectation \( E[\mu^{(b)}(S)] \) and variance \( \text{Var}[\mu^{(b)}(S)] \) converge to their asymptotic expressions at rates \( O(m^{-1}) \) and \( O(m^{-3/2}) \), respectively.

We next state a heterogeneous multivariate extension to the asymptotic granulometric mixing theorem \[17\]. The original (long) proof goes through with some minor changes. In stating the theorem, we let \( m = m_1 + m_2 + \cdots + m_d \) be the total sample size, \( u \) be the numerator in Eq. (18) divided by \( m, v \) be the denominator divided by \( m, H(u, v) = u/v = \mu^{(b)}(S), \) and \( \text{Eu} \) and \( \text{Ev} \) be the expectations \( E[u] \) and \( E[v] \) of \( u \) and \( v \), respectively.

**Theorem 2.** Let \( S \) be a random set of the form given in Eq. (17) for which the random grain-sizing variables \( r_{ij} \) are independent, each \( r_{ij} \) is selected from a sizing distribution \( P_i \) possessing moments up to order \( k + 2 \), the counts \( m_1, m_2, \ldots, m_d \) occur in known fixed proportions \( \theta_i = m_i/m, i = 1, 2, \ldots, d \), there exists a bound \( C \), independent of \( m \), and \( q > 0 \) such that \( H^q \leq C m^q \) for \( n > 1 \), and \( H \) has first and second derivatives and its second derivatives are bounded by a constant independent of \( m \) in a neighborhood of \( (\text{Ev}, \text{Ev}) \). Then, for the \( (\mathfrak{f}, \mathfrak{h}, \mathfrak{t}) \)-granulometry of Eq. (3) the distribution of \( H = \mu^{(b)}(S) \) is asymptotically normal with mean and variance given by

\[
E[H] = H(\text{Ev}, \text{Ev}) + O(n^{-1}),
\]

\[
\text{Var}[H] = \left( \frac{\hat{c}H}{\hat{c}u} (\text{Ev}, \text{Ev}) \right) \text{Var}[u] + 2 \hat{c}H \text{EvCov}[u, v] + \left( \frac{\hat{c}H}{\hat{c}v} (\text{Ev}, \text{Ev}) \right)^2 \text{Var}[v] + O(n^{-3/2}).
\]

(Note that \( \text{Eu} \) and \( \text{Ev} \) are calculated via the expected value of transformed variables.)

To illustrate Theorem 2, consider the heterogeneous scaling function \( h^{-1}(t) = t^{l/2} \) and the single-primitive random set of Eq. (23) governed by a gamma distribution having parameters \( x \) and \( \beta \). With the choice of a gamma sizing distribution, computation of the asymptotic expression for the expected value of the pattern-spectrum mean in fractional cases is possible. In this case we require \( E[X^{2.5}] \) for the numerator (Eq. (24)) of the asymptotic expression for the pattern-spectrum mean. (Note that this expectation does not exist in the case of Gaussian sizing distributions since fractional exponents applied to the negative part of the range yield imaginary values.) The fractional-power moment is evaluated as

\[
\mu^{(2.5)} = C \int_0^\infty x^{2.5} e^{-x/\lambda} \, dx,
\]

where \( C \) is the normalization constant for the probability integral. Applying the identity \( \Gamma(x + 1) = x\Gamma(x) \) twice
and using the fact that $\Gamma(0.5) = \pi^{1/2}$, we find that for large $N$,

$$E[\mu^{(1)}] \approx (\pi \beta)^{0.5} \frac{\beta + 1.5 \beta + 0.5 \cdots (0.5)}{\beta + 1}.$$  \hspace{2cm} (27)

A simulation with 100 images having a mean number of 50 circular grains governed by a gamma sizing distribution with $\alpha = 10$ and $\beta = 1$ produced 3.29 for the average pattern-spectrum mean, the exact value predicted by Eq. (27).

5. Heterogeneous granulometric classification

Univariate heterogeneous granulometries can achieve greater discrimination than univariate homogeneous granulometries without the increasing computational cost of multivariate granulometries. Given a set of structuring elements, the path defining a univariate heterogeneous granulometry can be selected to increase class separation among textures. This principle can be discerned with the aid of Fig. 5, which depicts bivariate

![Fig. 5. Bivariate size densities corresponding to three different textures and granulometric analysis employing two structuring elements.](image)

(a) (b) (c)

![Fig. 6. Synthetic textures.](image)
size densities corresponding to three different textures and granulometric analysis employing two structuring elements. Paths 1 and 2 correspond to the marginal single-structuring-element granulometries. Path 3 extracts the univariate homogeneous granulometry involving both structuring elements. Path 4 corresponds to a heterogeneous scaling. The pattern spectra corresponding to paths 1–4 are depicted in the figure. Examination of the pattern spectra profiles reveals the superiority of the heterogeneous approach. The spectra of all three

Fig. 7. Real and binarized real textures.
texture classes are clearly separated without any overlap for the profile generated by path 4. All other granulometries experience substantial amount of inter-class overlap which will produce features with lower discriminating power.

To illustrate the discriminative power of heterogeneous granulometries with respect to slight texture variations when using vertical and horizontal linear structuring elements, we employ the same synthetic (Fig. 6) and binarized real textures (Fig. 7) that we have previously used for multivariate granulometric classification [14]. The synthetic textures have been generated with primitives that produce strongly overlapping pattern spectra. Fifteen parameterized ellipsoidal curves are used as heterogeneous paths (Fig. 8). These are discretized for digital application (Fig. 9). Heterogeneous pattern spectra are derived from sampling the multivariate granulometric size distributions along these paths. A large number of fractional moments of the univariate pattern spectra are extracted into feature vectors to obtain texture representations as complete as possible. Later the granulometric fractional moments associated with each path are projected into a compressed feature set via the Karhunen-Loeve transform. The transformed features are used for training and classification. For the synthetic textures, the first heterogeneous path above the $\mathfrak{B}$-granulometry (diagonal) had the highest classification, 96.0%. In this case, the $\mathfrak{B}$-granulometry also did fairly well with a classification rate of 94.7%. In both cases, the best classification is achieved with the same number of features. At such relatively high classification rates, an improvement of 1.3% is significant. There was extensive performance variation across different paths. The peak performance goes as low as 80% for both of the marginal granulometries. For the real images, almost perfect classification is achieved using the first path below the diagonal, and again the marginal granulometries were among the poorest performers. The results clearly suggest that heterogeneous granulometries can extract more information from a texture process than its traditional counterparts.

6. Conclusion

Heterogeneous granulometries form an extended class of granulometries based on nonlinear scaling. As such they provide greater flexibility for the formation of sieving filters and texture classification. In particular, univariate heterogeneous granulometries can provide better classification than univariate homogeneous granulometries without incurring the increased computational cost of multivariate granulometries. There is a straightforward relationship between homogeneous multivariate orthogonal granulometries and heterogeneous univariate orthogonal granulometries. Both the representational and asymptotic mixing theorems for homogeneous granulometries extend to heterogeneous granulometries.

The mixing theory applies to the disjoint grain model of Eq. (17). A natural question concerns the status of the mixing theory when there is grain overlapping. One way to interpret the issue is in terms of robustness. Specifically, suppose there is grain overlapping and the image is
segmented so that the new components “approximate” the original grains prior to overlapping. If granulometric computations are made on the segmented image, to what degree does the representation of Eq. (18) remain valid? To wit, if \( S \) is the segmented image and \( S_0 \) is an image formed as a disjoint union of the grains whose union forms the unsegmented image, then can we say something about the difference between \( \mu^{(k)}(S) \) and \( \mu^{(k)}(S_0) \)? A tight quantification relating to this difference in the context of heterogeneous granulometries has yet to be discovered. However, we can look to the univariate homogeneous theory to see the kind of result that we might be able to achieve. Under practical overlap constraints and a suitable segmentation procedure, for the single-structuring element granulometry \( S \circ tB \), there exist lower and upper bounds \( \omega_1 \) and \( \omega_2 \), which are dependent on the degree, \( \delta \), of overlap, such that \( \omega_1 \leq \mu^{(k)}(S) \leq \omega_2 \). For certain models, \( \omega_1, \omega_2 \rightarrow \mu^{(k)}(S_0) \) as \( \delta \rightarrow 0 \) (no overlap) [19].

**Appendix A**

We prove Theorem 1. The Euclidean property of an ordinary opening, \( rA \ast B = r(A \ast B) \), together with the fact that area is homogeneous of degree 2, leads to

\[
\Omega_3(h(t)) = \sum_{i=1}^{d} \sum_{j=1}^{m} r_{ij}^{2} \Phi_A(h_1(t_1)/r_{ij}, \ldots, h_n(t_n)/r_{ij})
\]  

(28)

where \( h(t) = (h_1(t_1), \ldots, h_n(t_n)) \). Applying the identity

\[
-\Omega_a(h(t)/r_{ij}) = \alpha(A)[\Phi_A(h(t)/r_{ij}) - 1],
\]

(29)

to the definition of \( \Phi_A \) along with algebraic manipulation yields

\[
\Phi_2(h(t)) = \frac{\sum_{i=1}^{d} \alpha(A) \sum_{j=1}^{m} r_{ij}^{2} \Phi_A(h_1(t_1)/r_{ij}, \ldots, h_n(t_n)/r_{ij})}{\sum_{j=1}^{m} \alpha(A) \sum_{j=1}^{m} r_{ij}^{2}}.
\]

(30)

Hence, the kth moment of the pattern spectrum is

\[
\mu^{(k)}(S) = \frac{\sum_{i=1}^{d} \alpha(A) \sum_{j=1}^{m} r_{ij}^{2} \Phi_A(h_1(t_1)/r_{ij}, \ldots, h_n(t_n)/r_{ij})}{\sum_{j=1}^{m} \alpha(A) \sum_{j=1}^{m} r_{ij}^{2}} = \sum_{i=1}^{d} \alpha(A) \sum_{j=1}^{m} r_{ij}^{2} \Phi_A(h_1(t_1)/r_{ij}, \ldots, h_n(t_n)/r_{ij}).
\]

(31)

If \( h_i(t_i) = t_i^p \), where \( p_i \) is a positive integer, then the preceding expression reduces to

\[
\mu^{(k)}(S) = \frac{1}{p_1 \cdots p_n} \sum_{j=1}^{m} \alpha(A) \mu(A) \delta^{(k_1 + 1 - p_1)/(p_1) + \cdots + (k_n + 1 - p_n)/(p_n)} \sum_{j=1}^{m} r_{ij}^{2} \delta^{(k_1 + 1)/p_1 + \cdots + (k_n + 1)/p_n}
\]

(32)

which is the desired result.

The change of variables used in Eq. (31) requires some justification. Eq. (31) states that

\[
E[(r_1^1 \cdots r_n^1)] = E[(h_1^{-1}(r_1s_1))^k \cdots (h_n^{-1}(r_ns_n))^k].
\]

(33)

in their respective probability spaces. Because \( A_i \) is compact, for each \( i \) the multiple Stieltjes integral involves a bounded integrand over a compact set. Let \( \Xi = \Xi(A, A, P) \) and \( \Xi = \Xi(A', A', P') \) be the probability spaces associated with the distribution functions \( \Phi_A(t_1, \ldots, t_n) \) and \( \Phi_A(h_1(t_1)/r_{ij}, \ldots, h_n(t_n)/r_{ij}) \). Hence, the measure \( \Omega \) and \( \Omega_k \) coincide with the domains of their respective distribution functions on \( \mathbb{R}^n \), \( \Omega_k \) and \( \Omega_k \) are the compact regions shown in Fig. 10 (due to the constraints on \( h_i(t_i) \)). The random vectors \( \mathbf{X} = (X_1, \ldots, X_n) \) and \( \mathbf{X}' = (X'_1, \ldots, X'_n) \) associated with these probability spaces are obtained via the identity mappings

\[
(\Omega, A, \mathbf{X}) = (\mathbb{R}^n, B_{\mathbb{R}^n}) \rightarrow (\Omega, A, \mathbf{X}').
\]

(34)

Where \( B \) denotes the Borel field. The associated distribution functions \( F \) and \( F' \) are the images of \( P \) and \( P' \) under the equivalence class of these mappings. Let \( \phi(t_1, t_2, \ldots, t_n) \) be the measurable mapping between the probability spaces \( \Xi \) and \( \Xi' \) defined by \( \phi(t_1, t_2, \ldots, t_n) = (h_1(t_1), h_2(t_2), \ldots, h_n(t_n)) \). For any elementary \( n \)-dimensional rectangular event \( E' \) on \( \Xi' \), there is associated a rectangular event \( E \) on \( \Xi \) such that \( \Delta F(R) = \Delta F'(R) \). Referring to Fig. 10, it is clear that

\[
\Delta F'(R') = F'(a') - F'(b') = F(d') - F(d)
\]

(35)

Therefore, \( \phi : \Xi \rightarrow \Xi' \) is a morphism between the probability spaces \( \Xi \) and \( \Xi' \). Consequently, for any integrable function \( g \) on \( \Xi' \), \( \phi^*g \) is integrable on \( \Xi \) and \( E[\phi^*g] = E[g] \), where the expectations are taken in the corresponding measure spaces. Hence, the measure \( dF \) can be related to \( dF' \) via the Jacobian of \( \phi(t_1, t_2, \ldots, t_n) \) as \( dF = dF'/J \). This is the content of the second equality of Eq. (31).
Fig. 10. Mappings for morphism.

References


About the Author—SINAN BATMAN holds a B.Sc., 1990, in electrical engineering from Middle East Technical University, Ankara, Turkey, and a Ph.D., 1998, in electrical engineering from Texas A&M University, specializing in nonlinear stochastic signal processing. He is currently an associate research scientist in the Center for Imaging Science at the Electrical and Computer Engineering Department of The Johns Hopkins University. His current areas of interest are the development of nonlinear techniques in the areas of biomedical image processing and automated mine detection. He has published several journal articles and conference proceedings in the areas of nonlinear image processing, stochastic pattern recognition, and optimization theory.

About the Author—EDWARD R. DOUGHERTY holds an M.S. in computer science from Stevens Institute of Technology and a Ph.D. in mathematics from Rutgers University. He is currently a Professor in the Department of Electrical Engineering at Texas A&M University. He is editor of the SPIE/IS&T Journal of Electronic Imaging and of the SPIE/IEES Series on Imaging Science and Engineering. He is the author of eleven books, editor of four books, and has published numerous papers in nonlinear filtering and mathematical morphology. His current interest is the optimal design of nonlinear filters, granulometric analysis, and informatics for dCNA microarrays.
About the Author—FRANCIS SAND is an Associate Professor in Fairleigh Dickinson University’s School of Computer Science and Information Systems in New Jersey. He holds a Ph.D. in Mathematics from Princeton University, an M.S. (Applied Math.) from N.Y. Polytechnic University and a B.Sc. (Physics, Math.) from Cape Town University in South Africa. He has published extensively in statistics, operations research, systems theory and mathematical morphology. His current research interests include development of a practical approach to robust filter design for a wide class of images. He is a leader in the use of computers for distance learning at FDU and is currently teaching a graduate course in computer science via the internet.