Optimization of Linear Filters Under Power-Spectral-Density Stabilization

Artyom M. Grigoryan, Member, IEEE, and Edward R. Dougherty

Abstract—Geometric-mean filters compose a family of filters indexed by a parameter $k$ varying between 0 and 1. They have been used to provide frequency-based filtering that mitigates the noise suppression of the optimal-linear Wiener filter in the blurred-signal-plus-noise model. For $k = 0$ and $k = 1$, the geometric-mean filter gives the inverse filter and the Wiener filter for the model, respectively. The geometric mean for $k = 1/2$ has previously been derived as the optimal linear filter for the model under power-spectral-density (PSD) equalization. This constraint requires the PSD of the filtered signal to be equal to the PSD of the uncorrupted signal that it estimates. This paper defines the notion of PSD stabilization, in which the PSD of the restored signal is equal to a predetermined function times the PSD of the uncorrupted signal. A particular parameterized stabilization function yields the geometric-mean family as the optimal linear filter for the model under PSD stabilization. Relative to unconstrained optimization, geometric means are suboptimal; however, we consider a parameterized model for which the noise is such that the geometric-mean filters provide optimal linear filtering. In the altered signal-plus-noise model for which the geometric mean is optimal, the blurring function resulting from an imaging system; additive noise.

I. INTRODUCTION

The geometric-mean filter is a parameterized class of frequency-based linear filters used to restore images suffering from blur and additive noise [1], [2]. A limiting condition for the parameters results in the Wiener filter for the degradation model typically associated with the geometric mean. As classically introduced, the geometric mean is based on heuristic considerations that attempt to mitigate detrimental effects of the Wiener filter, especially in regard to more than desirable suppression of high frequencies by the Wiener filter. For a particular parametric choice, the geometric-mean filter takes a special form that was originally derived via a homomorphic approach [3] and was subsequently derived via minimization of mean-square error under power-spectral-density equalization, which means that the power spectral densities (PSDs) of the ideal and restored signals must be equal [1]. This paper discusses a more general constrained optimization in which geometric-mean filters result from a linear model in which filter optimization is constrained by a stabilization factor between the PSDs of the ideal and restored signals. Moreover, we will consider an altered model in which geometric-mean filters provide optimal linear filtering for a parameterized linear model closely related to the original. It will be seen that the kind of optimal filtering provided by geometric means has precisely the nature as that desired by the heuristic considerations, leading to its original formulation. Since the model under which geometric-mean filters are optimal is parameterized (thereby leading to a parameterized family of filters), a robustness question arises: What is the effect of applying the geometric mean for a specific model parameter when a different model parameter describes the state of nature? The robustness problem is inherently Bayesian and will be treated as such.

To maintain a smooth flow throughout the paper, we will defer to the Appendix for expanded derivations of equations when these are necessary.

In the spatial domain, the restoration problem typically associated with the geometric mean is the blurred-signal-plus-noise model described the integral equation

$$\int_{\mathbb{R}^2} h(y - x) o(x) \, dx + n(y) = i(y), \quad x, y \in \mathbb{R}^2 \tag{1}$$

where

- $h(y)$ blurring function resulting from an imaging system;
- $o(x)$ signal to be estimated;
- $n(y)$ additive noise.

and we desire an optimal estimator $\hat{o}(x)$ of the true image $o(x)$ from the given data $i(y)$ by means of a linear filter. In the frequency domain, the problem is posed as finding an estimator $\hat{O}(\omega)$ for $O(\omega)$, where

$$H(\omega)O(\omega) + N(\omega) = I(\omega) \tag{2}$$

and where uppercase letters indicate the Fourier integrals of the respective functions. Here, we note that $H(\omega)$ is an ordinary Fourier transform but that the other three functions are Fourier integrals of random functions. For discrete signals, there is no difficulty here, but for random analog signals, these integrals are white noise processes and must be handled with some care.
A linear filter $Y(\omega)$ will be a multiplicative operator yielding $O(\omega) = Y(\omega)I(\omega)$.

Following the original spectral approach taken by Hellstrom [6], we define the mean-square error in the frequency domain by

$$\rho^2(\hat{O}, O) = \mathbb{E}\left[ \int_{\mathbb{R}^2} |\hat{O}(\omega) - O(\omega)|^2 d\omega \right] = \int_{\mathbb{R}^2} \mathbb{E}[|\hat{O}(\omega) - O(\omega)|^2] d\omega.$$ \hspace{1cm} (3)

The error is defined as the expected $L_2$ error across the frequency domain. Assuming the stochastic integral in the first line exists in the mean-square sense, the $L_2$ error is itself a random variable, and the existence of the integral permits us to bring the expectation inside the integral so that the integral becomes deterministic. Let $\phi_O(\omega) = \mathbb{E}[|O(\omega)|^2]$ (up to the coefficient $2\pi$) and $\phi_N(\omega) = \mathbb{E}[|N(\omega)|^2]$ be the PSDs for the object and noise images, respectively, and let $\phi_{CN}(\omega) = \phi_N(\omega)/\phi_O(\omega)$ be the noise-to-signal ratio (NSR). The error of the $O(\omega)$ restoration by a linear filter $Y(\omega)$ is

$$\rho^2(\hat{Y}(\omega)) = \int_{\mathbb{R}^2} [1 - Y(\omega)H(\omega)]^2 \phi_O(\omega) + [Y(\omega)\phi_N(\omega)] d\omega.$$ \hspace{1cm} (4)

Via a Lagrange–Tikhonov technique [7], Hellstrom showed that the optimal linear filter has the transfer function of the classical Wiener filter

$$Y_{Wm}(\omega) = \frac{H(\omega)}{|H(\omega)|^2 + \phi_{CN}(\omega)}, \quad \omega \in \mathbb{R}^2 \hspace{1cm} (5)$$

where $H(\omega)$ is the complex conjugate of $H(\omega)$. The optimal linear filter depends on both the object signal and degradation. The error for the optimal linear filter is

$$\min_{Y \in Y} \rho^2(\hat{O}, O) = \int_{\mathbb{R}^2} \frac{\phi_O(\omega) \phi_N(\omega)}{|H(\omega)|^2 + \phi_{CN}(\omega)} d\omega. \hspace{1cm} (6)$$

For optimization relative to (3), the requirement that the signal and signal estimate possess the same PSD $\phi_O(\omega) = \phi_O(\omega)$ is called optimization under power spectrum equalization. Using methods of discrete linear algebra [1], Andrews and Hunt have shown that relative to a minimum mean-square-error optimization, an optimal linear filter for blurred-signal plus uncorrelated additive noise is given by the transfer function

$$Y_{hom}(\omega) = \left[ \frac{H(\omega)}{|H(\omega)|^2 + \phi_{CN}(\omega)} \right]^{1/2} \left[ \frac{1}{H(\omega)} \right]^{1/2} \hspace{1cm} (7)$$

We have denoted the filter by $Y_{hom}$ because the filter has been associated with homomorphic filtering [8]. Indeed, Stockham $et$ al. arrived at the same filter by a homomorphic approach [3]. This filter does not achieve minimum MSE owing to PSD equalization; however, the constraint is rather natural if our main concern is with the PSD.

II. OPTIMIZATION UNDER PSD STABILIZATION

The equalization constraint is generalized by requiring

$$\phi_O(\omega) = \alpha(\omega)\phi_O(\omega) \hspace{1cm} (8)$$

where $\alpha(\omega)$ is a positive stabilization function. Using a Lagrange–Tikhonov method, we determine the optimal solution for (2) under the PSD stabilization of (8). It is well known that the PSD of the observed image is

$$\phi_T(\omega) = |H(\omega)|^2 \phi_O(\omega) + \phi_N(\omega).$$ \hspace{1cm} (9)

The optimal solution of the image restoration under PSD stabilization is determined by the smoothing functional (Lagrangian)

$$\mathcal{L}_\alpha[\hat{Y}, O] = \int_{\mathbb{R}^2} \mathbb{E}[|O(\omega) - \hat{O}(\omega)|^2] d\omega + \int_{\mathbb{R}^2} \lambda(\omega)[\phi_O(\omega) - \alpha(\omega)\phi_O(\omega)] d\omega \hspace{1cm} (10)$$

where the real parameter $\lambda(\omega)$ is defined from (8). We will show that the response function of the optimal filter for this model, which we call optimization under power spectrum stabilization, is

$$Y_{\alpha}(\omega) = \sqrt{v(\omega)}v(\omega) Y_{Wm}(\omega) \hspace{1cm} (11)$$

where $v(\omega) \leq 1$ is the positive function defined as

$$v(\omega) = \frac{|H(\omega)|^2}{|H(\omega)|^2 + \phi_{CN}(\omega)} \hspace{1cm} (12)$$

and the inverse filter is given by $Y_{Wm}(\omega) = H(\omega)^{-1}$.

To arrive at (11), note that the Lagrangian in the integral (10) takes the form

$$\mathcal{L}_\alpha = \phi_O(\omega) + Y_{\alpha}(\omega)\mathbb{E}[|O(\omega)H(\omega) + N(\omega)|^2] - 2\phi_O(\omega) \mathbb{E}[Y_{\alpha}(\omega)H(\omega) + N(\omega)] + \lambda(\omega) Y_{\alpha}(\omega) \mathbb{E}[|O(\omega)H(\omega) + N(\omega)|^2] - \alpha(\omega)\phi_O(\omega)$$

where $\mathbb{E}[O(\omega)N(\omega)] = 0$ because $O(\omega)$ and $N(\omega)$ are uncorrelated. From the minimizing condition

$$\frac{\partial \mathcal{L}_\alpha}{\partial Y_{\alpha}} = Y_{\alpha}(\omega)\mathbb{E}[|O(\omega)H(\omega) + N(\omega)|^2](1 + \lambda(\omega))$$

$$- \phi_O(\omega) H(\omega) = Y_{\alpha}(\omega)[|H(\omega)|^2\phi_O(\omega) + \phi_N(\omega)](1 + \lambda(\omega))$$

$$- \phi_O(\omega) H(\omega) = 0 \hspace{1cm} (14)$$

we obtain the filter with the response function

$$Y_{\alpha}(\omega) = \frac{1}{1 + \lambda(\omega)} \frac{|H(\omega)|^2\phi_O(\omega) + \phi_N(\omega)}{1 + \lambda(\omega)Y_{Wm}(\omega)} \hspace{1cm} (15)$$
To find the parameter $\lambda(\omega)$, we use the stabilization constraint to obtain

$$\alpha(\omega)\phi_O(\omega) = \phi_O(\omega) = [Y_\alpha(\omega)]^2[H(\omega)]^2\phi_O(\omega) + \phi_N(\omega)$$

(16)

which together with (15) yields

$$\alpha(\omega)\phi_O(\omega) = \left[\frac{1}{1 + \lambda(\omega)}\right]^2 \left(\frac{H(\omega)\phi_O(\omega)}{[H(\omega)]^2\phi_O(\omega) + \phi_N(\omega)}\right)\cdot H(\omega)\phi_O(\omega).$$

(17)

Hence

$$\left(\frac{1}{1 + \lambda(\omega)}\right)^2 Y_\alpha(\omega) H(\omega) = \alpha(\omega),$$

(18)

Substituting the expression for $\lambda(\omega)$ into (15) yields

$$Y_\alpha(\omega) = \frac{1}{H(\omega)} \sqrt{\frac{[H(\omega)]^2}{[H(\omega)]^2 + \phi_N(\omega)}} \alpha(\omega)$$

(19)

which coincides with (11).

From (11), the restored image has the form

$$\hat{\omega}_\alpha(\omega) = \sqrt{\alpha(\omega)} \sqrt{v(\omega)} Y_{\text{inv}}(\omega) I(\omega).$$

(20)

If $\alpha(\omega) = \alpha_{\text{inv}}(\omega) = v(\omega)$, then the Wiener filter results. If $\alpha(\omega) = \alpha_{\text{inv}}(\omega) = v(\omega)^{-1}$, we obtain the inverse filter. Note that

$$\alpha_{\text{inv}}(\omega) - \alpha_{\text{inv}}(\omega) = v(\omega)^{-1} - v(\omega)$$

$$= \frac{\phi_O(\omega)}{[H(\omega)]^2 + \phi_N(\omega)} \frac{\phi_N(\omega)}{[H(\omega)]^2}$$

(21)

characterizes a degree of difference between the inverse and Wiener filters. The filters differ considerably at frequencies with the small values of the SNR $\phi_O/\phi_N(\omega)$ and response function $H(\omega)$. If $\alpha(\omega) \equiv 1$, then there is PSD equalization, and we obtain

$$Y_\alpha(\omega) = \sqrt{v(\omega)} Y_{\text{inv}}(\omega) = \sqrt{Y_{\text{inv}}(\omega)} \sqrt{Y_{\text{inv}}(\omega)} = Y_{\text{inv}}(\omega).$$

(22)

A particular stabilization function yields the geometric-mean family. Let

$$\alpha_k(\omega) = \alpha_{\text{inv}}(\omega)^{1-k} \alpha_{\text{inv}}(\omega)^{k} = v(\omega)^{1-k}$$

(23)

for $k \in [0, 1]$. Then

$$Y_k(\omega) = \left[\frac{H(\omega)}{[H(\omega)]^2 + \phi_N(\omega)}\right]^k \left[\frac{1}{H(\omega)}\right]^{1-k},$$

(24)

In particular, $\alpha_0(\omega) = \alpha_{\text{inv}}(\omega)$, $\alpha_1(\omega) = \alpha_{\text{inv}}(\omega)$, and $\alpha_k(\omega) = \alpha_{\text{inv}}(\omega)$.

If we consider the special case of deblurring, meaning that there is no additive noise, then the solution of (11) reduces to

$$Y_\alpha(\omega) = \sqrt{\alpha(\omega)} Y_{\text{inv}}(\omega).$$

Owing to the requirement of PSD stabilization, the filter is suboptimal relative to the inverse filter $Y_{\text{inv}}(\omega)$, which is optimal when there is no additive noise. In the case of the geometric mean filter, the absence of additive noise means that $\alpha_k(\omega) = 1$ and $Y_k(\omega) = Y_{\text{inv}}(\omega)$. A number of papers have addressed restoration of blurred images, including [9]–[12].

In accordance with (11), two different stabilization functions $\alpha(\omega)$ and $\beta(\omega)$ yield two different filters $Y_\alpha(\omega) = \sqrt{v(\omega)\alpha(\omega)} Y_{\text{inv}}(\omega)$ and $Y_\beta(\omega) = \sqrt{v(\omega)\beta(\omega)} Y_{\text{inv}}(\omega)$, yielding different errors. The difference in the errors is given by

$$\delta(\alpha, \beta) = \rho^2(\hat{\omega}_\beta, O) - \rho^2(\hat{\omega}_\alpha, O)$$

$$= \int_{\mathbb{R}} \left(\sqrt{\beta(\omega)} - \sqrt{\alpha(\omega)}\right) v(\omega) \phi_0(\omega) \phi_N(\omega) - 2\phi_0(\omega)$$

$$\cdot \left[\sqrt{\beta(\omega)} - \sqrt{\alpha(\omega)}\right] \sqrt{v(\omega)} d\omega.$$

(25)

In particular, for the one-parameter geometric-mean filters with response functions $Y_\alpha(\omega)$ and $Y_\beta(\omega)$, $a, b \in [0, 1]$, we have $\sqrt{v(\omega)\alpha(\omega)} = v^a(\omega)$, $\sqrt{v(\omega)\beta(\omega)} = v^b(\omega)$, and

$$\delta(a; b) = \int_{\mathbb{R}} [v^b(\omega) - v^a(\omega)] [v^{a-1}(\omega) + v^{a-1}(\omega) - 2]$$

$$\cdot \phi_0(\omega) d\omega.$$

(26)

Since $a, b \leq 1$

$$v^{a-1}(\omega) + v^{a-1}(\omega) = (v^{-1}(\omega))^{1-b} + (v^{-1}(\omega))^{1-a} \geq 2$$

(27)

and $\delta(a; b) > 0$ if $b < a$.

III. GEOMETRIC MEAN AS AN OPTIMAL LINEAR FILTER

In general, the optimal linear filter under PSD stabilization is not optimal for the linear model of (2) for which it has been derived. In particular, for $k \neq 1$, the geometric mean is not optimal. However, $Y_k(\omega)$ is optimal for any model for which it is the Wiener filter. Assuming a fixed imaging system (fixed $H$), we need determine a noise PSD $\phi_N(\omega)$ such that $Y_k(\omega)$ is the Wiener filter for the model

$$H(\omega)\phi_0(\omega) + N_k(\omega) = I_k(\omega).$$

(28)

For $Y_k(\omega)$ to be optimal for this model, the following equation must be satisfied:

$$\left(\frac{H(\omega)\phi_0(\omega)}{[H(\omega)]^2 + \phi_N(\omega)}\right)^k \left(\frac{1}{H(\omega)}\right)^{1-k}$$

$$= \frac{H(\omega)\phi_0(\omega)}{[H(\omega)]^2 + \phi_N(\omega)},$$

(29)
Solving for $\phi_{N_k}(\omega)$ yields
\[
\phi_{N_k}(\omega) = \left[ |H(\omega)|^2 \phi_O(\omega) \right]^{1/k} = \left[ |H(\omega)|^2 \phi_{N}(\omega) \right]^{1/k},
\]
with this noise PSD, the geometric-mean filter $Y_k(\omega)$ is the optimal linear filter for the model of (28).

The Wiener filter solution for (2) occurred under the assumption that the signal and noise are uncorrelated. To examine the relationship between the models of (2) and (28), let us assume that $N$ is white. Then, unless $k = 1$, in which case the two models agree, $N_k$ is not white. The PSD of $N_k$ is the difference between the $k$th powers of the PSD for $I(\omega)$, the output of the original observation model, and $I(\omega)$, which is the output of the original model, absent the noise, or simply the output of the noiseless imaging system. Hence, although the model of (28) still assumes uncorrelated signal and noise, it does not assume that the noise characteristics are unrelated to the imaging system. The noise $N_k$ is modeled via white noise, but in the case of (28), the model involves a nonlinear (fractional power) difference between the noisy imaging system and the nonnoisy imaging system. Quoting Andrews and Hunt on the effect of geometric-mean filters [relative to the original model of (2)],[1] “The motivation for such a parameterization is the desire to de-emphasize the low-frequency dominance of the Wiener filter while avoiding the early singularity of the inverse filter.” This effect is clearly seen for the model of (28) by simply writing
\[
Y_k(\omega) = \frac{H^2(\phi_O + \phi_{N}(\omega)) - |H|^2 \phi_{N}(\omega)}{|H|^2 \phi_O + |H|^2 \phi_N(\omega)},
\]
for $0 \leq k < 1$, as a Wiener filter [for the model of (28)], the geometric-mean filter provides less frequency attenuation than does the Wiener filter for the original model.

The geometric mean is more generally defined by replacing $\phi_N(\omega)$ by $\gamma \phi_N(\omega)$ in (9), where $0 \leq \gamma \leq 1$. This yields a doubly parameterized geometric mean $Y_k(\omega)$. If we change the model of (2) by replacing $N(\omega)$ with $\sqrt[\gamma]{N(\omega)}$, then the preceding analysis goes through with $\gamma \phi_N(\omega)$ in place of $\phi_N(\omega)$. It means that (5) applies with $\phi_{N/O}(\omega) = \phi_{N}(\omega)/\phi_O(\omega)$ in place $\phi_{N/O}(\omega)$. Because $0 \leq \gamma \leq 1$, (32) continues to apply but with $\phi_{N}(\omega) = \gamma \phi_N(\omega)$ in place of $\phi_N(\omega)$.

IV. ROBUSTNESS OF THE GEOMETRIC MEAN RELATIVE TO OPTIMIZATION

Since the model of (28) is parameterized by $k$, a robustness question arises: What increased error occurs if we employ the optimal filter for noise $N_k$ when the correct model has noise $N_a$? Qualitatively, a filter is said to be robust when its performance degradation is acceptable for images statistically close to those for which it has been designed. The problem of robustness arises because optimization is relative to an image model. In the case of geometric-mean filters, optimality depends on the noise parameter in (28). If, for instance, we assume that $k = 1$, and thereby apply the original Wiener filter for (2), and the actual value of the parameter is $k = a < 1$, then there is an increase in error over that which would occur by applying $Y_a(\omega)$.

The classical approach to studying robustness in the context of linear filters is to try to find a filter whose maximum error over all values of the model parameter will be minimal among all optimal linear filters derived across the values of the model parameter [13]–[16]. This approach results in finding, when possible, a solution to a mini-max error criterion depending on the PSDs of the observed and ideal signals. Relative to our setting, in which the state of nature corresponds to the PSD for the noise, the mini-max approach considers a class of noise PSDs $\{\phi_{N_a^*} : a \in A\}$, and the problem is to find a value $a^*$ of the parameter $a$ satisfying
\[
a^* = \arg \min_{a \in A} \left\{ \max_{\phi \in \Phi} \rho^2(\hat{O}_{a,b}, O) \right\}
\]
where $\hat{O}_{a,b}$ is the restored image for noise in state $a$ using the optimal linear filter $Y_b(\omega)$ for noise state $b$. If the preceding equation can be solved, then the mini-max robust filter is the optimal filter for noise $N_a^*$. Since they are chosen conservatively, when they exist, mini-max robust filters may perform poorly for likely values of the parameter.

Here, we will take a Bayesian approach to robustness: one that has been applied for optimal digital binary filters [17]. We assume there is a prior distribution (density) $f(\alpha)$ for the parameter, and we consider robustness relative to this prior distribution. Intuitively, we are most concerned that the filter perform well over those states of nature $\alpha$, where the probability mass of the distribution describing the occurrence of the states of nature is concentrated.

The robustness of the optimal filter relative to the model of (28) is the increase in error due to using the filter $Y_b(\omega)$ for the noise $N_a$:
\[
\kappa(\alpha; b) = \rho^2(\hat{O}_{a,b}, O) - \rho^2(\hat{O}_{a,a}, O) = \rho^2(Y_b I_0, O) - \rho^2(Y_a I_0, O) = \rho^2(Y_b (H_O + N_a), O) - \rho^2(Y_a (H_O + N_a), O)
\]
\[
= \int_{\mathcal{C}} \left[ \phi^b(\omega) - \phi^a(\omega) \right] \left[ \phi_{N}(\omega) + \frac{\phi_{N}(\omega)}{|H|^2} \right] - 2\phi_{N}(\omega) d\omega
\]
where $\phi_{N_a^*}(\omega)$ is defined by (30) with $k = a$. The mean robustness is the expected error increase from using $Y_b(\omega)$ instead of the actual state of nature. It is defined by the expectation of $\kappa(\alpha; b)$ relative to the density $f(\alpha)$:
\[
\kappa(b) = \mathbb{E}[\kappa(\alpha; b)] = \int_0^1 \left[ \rho^2(\hat{O}_{a,b}, O) - \rho^2(\hat{O}_{a,a}, O) \right] f(a) da
\]
since $\phi_{\alpha}(\omega) = |H(\omega)|^2 \phi_{\beta}(\omega) + \phi_{\eta_\alpha}(\omega)$. A maximally robust state is one for which $\kappa(b)$ is minimized [18].

According to the definition of robustness [see (35)], taking the expectation with respect to the distribution of $\alpha$ yields

$$
\mathbb{E}[\rho^2(\hat{\alpha}, b, O)] = \kappa(b) + \mathbb{E}[\rho^2(\hat{\alpha}, a, O)].
$$

(36)

Hence, if the actual state of nature is unknown so that the error $\rho^2(\hat{\alpha}, b, O)$ for the filter $Y_b(\omega)$ is a random variable depending on $a$, then the expected error is minimized by choosing $b$ to be a maximally robust state. Minimization of the mean robustness depends on the distribution of $a$. We refer to [18] for a discussion of such minimization.

In general, given two different stabilization functions $\alpha(\omega)$ and $\beta(\omega)$, the increase in error when applying the filter $Y_\beta(\omega)$ instead of $Y_\alpha(\omega)$ is

$$
\kappa(\alpha; \beta) = \int_{\mathbb{R}} \left[ (\beta(\omega) - \alpha(\omega)) \rho(\omega) \left( \phi_{\beta}(\omega) + \frac{\phi_{\eta_\alpha}(\omega)}{|H(\omega)|^2} \right) - 2\phi_{\beta}(\omega) \sqrt{\rho(\omega)} \left[ \sqrt{\beta(\omega)} - \sqrt{\alpha(\omega)} \right] \right] d\omega.
$$

(37)

To illustrate filter effects, we consider a discretized random Boolean function whose primary function is a pyramid. The model is strict-sense stationary. Fig. 1 shows

a) realization of the image;
b) realization after blurring;
c) blurred and additive-noised degraded realization;
d) degraded image restored by the Wiener filter.

Errors for parts b), c), and d) are 0.0082, 0.0092, and 0.0056, respectively. Note that for $b > 0.3$, the errors are not much greater than that for the Wiener filter for the model.

The results depicted in Fig. 2 are more fully explained in terms of the robustness function. We consider two practical cases relative to (28). If the true state of nature consists only of blur and no additive noise, meaning $a = 0$, and we use the filter $Y_0(\omega)$, then the robustness is defined by $\kappa(b; 0)$. If the true state of nature is given by (2), meaning that $a = 1$, and we use the filter $Y_1(\omega)$, then the robustness is defined by $\kappa(1; b)$. These robustness curves are shown in Fig. 3. $\kappa(0; b)$ and $\kappa(1; b)$ give the increases in error owing to PSD stabilization when the inverse and Wiener filters are optimal, respectively. For $b > 0.3$, $\kappa(1; b)$ is close to 0, which explains the phenomena of Fig. 2.
Fig. 2. Applying the optimal linear filters (a) $\Psi_{0.63}$, (b) $\Psi_{0.4}$, (c) $\Psi_{0.525}$, and (d) $\Psi_{0.82}$. Errors of restoration respectively equal 0.0273, 0.0077, 0.0063, and 0.0057.

Fig. 3. Robustness curves (a) $\kappa(1; b)$ and (b) $\kappa(1; b)$. 
V. Conclusion

PSD stabilization yields filters, or families of filters, that provide linear filtering for which optimality is constrained by some relationship between the PSDs of the restored and uncorrupted signals. A particular instance gives the geometric-mean family. By changing the PSD of the noise in the model, we can arrive at a model for which the geometric-mean family provides optimal linear filtering. Examination of this model shows that it naturally leads to the kinds of filtering effects for which the geometric-mean filters were originally heuristically designed. Relative to the model parameter, the geometric-mean filters are fairly robust as long as $k$ is not small. For both uniform and normal distribution of $k$, the original PSD-equalized geometric-mean filter is close to providing maximal robustness. Given these various observations, it is not surprising that the geometric-mean filter, especially the PSD-equalized filter, has proven useful.

Appendix

Expanded Equations

A1) Equation (4):

$$
\hat{r}^2(\hat{O}, O) = \mathbb{E}\left[ \int_{\mathbb{R}^2} |O(\omega) - Y(\omega)(O(\omega)H(\omega) + N(\omega))|^2 d\omega \right]
$$

A2) Equation (13):

$$
\lambda(\omega)[\phi(\omega) - a(\omega)\phi(\omega)]
$$

A3) Equation (16):

$$
\alpha(\omega)\phi(\omega) = \phi(\omega) + \mathbb{E}[\|O(\omega)H(\omega) + N(\omega)\|^2]
$$

A4) Equation (17):

$$
\alpha(\omega)\phi(\omega) = \frac{1}{1 + \lambda(\omega)} \mathbb{E}[\|O(\omega)H(\omega) + N(\omega)\|^2]
$$

A5) Equation (19):

$$
Y_{\alpha}(\omega) = \frac{1}{1 + \lambda(\omega)} Y_{W}(\omega) = \sqrt{\frac{\alpha(\omega)}{\mathbb{E}[\|O(\omega)H(\omega) + N(\omega)\|^2]}} Y_{W}(\omega)
$$

A6) Equation (22):

$$
Y_{\alpha}(\omega) = \sqrt{\mathbb{E}[\|O(\omega)H(\omega) + N(\omega)\|^2]} Y_{\text{inv}}(\omega)
$$

A7) Equation (24):

$$
Y_{\alpha}(\omega) = Y_{\text{inv}}(\omega) = \sqrt{\mathbb{E}[\|O(\omega)\|^2] - \sqrt{\text{det} \mathbb{K}(\omega)}} Y_{\text{inv}}(\omega)
$$
\( \delta(\alpha; \beta) = \int_{\mathbb{R}^2} \left( \frac{1}{H(\omega)} \right) \left( \frac{1}{H(\omega)} \right)^{i-k} \left( \frac{H(\omega)}{2\phi(\omega)} \right)^{i-k} \right) d\omega. \)

\( \delta(\alpha; b) = \int_{\mathbb{R}^2} \left( \frac{H(\omega)^2 \phi(\omega)}{H(\omega)^2 + \phi_N^2(\omega)} \right) \left( \frac{1}{H(\omega)} \right)^{i-k} \left( \frac{1}{H(\omega)} \right)^{i-k} \right) d\omega. \)

Equation (26):
\( \beta(\omega) = \nu^{2\alpha-1}(\omega), \quad \alpha(\omega) = \nu^{2\alpha-1}(\omega), \quad a, b \in [0, 1] \)

\( \delta(\alpha; b) = \int_{\mathbb{R}^2} \left( \frac{1}{H(\omega)^2} \phi(\omega) + \phi_N^2(\omega) \right) d\omega. \)

Equation (30):
\( \left( \frac{H(\omega)^2 \phi(\omega)}{H(\omega)^2 + \phi_N^2(\omega)} \right)^{i-k} \left( \frac{1}{H(\omega)} \right)^{i-k} \right) d\omega. \)
A11) Equation (32):
For positive numbers \(a\) and \(b\), we consider two functions \(f_1(x) = x^a\) and \(f_2(x) = (a + b)x - b\) for \(x \in [0, 1]\). Functions \(f_1(x)\) and \(f_2(x)\) are convex from below \(f''_1(x) \geq 0\) and \(f''_2(x) \geq 0\), monotonic increasing or decreasing functions with boundary conditions: \(f_1(1) = f_2(1) = a\) and \(f_1(0) = f_2(0) = 1 - b\). It is not difficult to see that the graphs of these functions do not have an intersection inside the interval \([0, 1]\). Therefore, \(f_1(x) \geq f_2(x)\), for all \(x \in [0, 1]\). That is, \(x^a \geq (a + b)x - b\).

A12) Equations (34) and (37):
\[
\kappa(Y_{\alpha}; Y_{\beta}) = \rho^2(Y_{W_{\alpha}}(HO + N_\alpha), O) - \rho^2(Y_{W_{\alpha}}(HO + N_\alpha), O)
\]
where, due to (4)
\[
\rho^2(Y_{W_{\alpha}}(HO + N_\alpha), O) = \int_{R^2} \left[ |Y_{W_{\alpha}}(\omega)|^2 - |Y_{W_{\alpha}}(\omega)|^2 \right] d\omega
\]
\[
\rho^2(Y_{W_{\alpha}}(HO + N_\alpha), O) = \int_{R^2} \left[ |Y_{W_{\alpha}}(\omega)|^2 - |Y_{W_{\alpha}}(\omega)|^2 \right] d\omega
\]

Therefore, similar to (25), we obtain
\[
\kappa(Y_{\alpha}; Y_{\beta}) = \int_{R^2} \left[ |(\beta(\omega) - \alpha(\omega))\tau(\omega)|H(\omega)|^{-2}\right]
\]
\[
\int_{R^2} \left[ \phi_\omega(\omega) + \phi_\omega(\omega) \right] d\omega
\]

since \(\phi_\omega(\omega) = |H(\omega)|^2\phi_\omega(\omega) + \phi_\omega(\omega).\)

For the geometric-mean filters, \(\beta(\omega) = \sqrt[2]{1}(\omega), \alpha(\omega) = \sqrt[2]{1}(\omega, a, b \in [0, 1]. and

\[
\kappa(\alpha; b) = \kappa(Y_{\alpha}; Y_{b})
\]

References